Strip Mining in the Abandoned Orefields of Nineteenth Century Mathematics

0. Apology

This was my 1984 (NYU) talk. My 1986 (Stanford) conference talk began

"The last time Dick [Jenks] asked me to do this, my talk was so weird, and my slides so complicated that I left much of my audience exhausted and confused. By way of remedy, we have amassed here today an array of video equipment so expensive that it can actually reduce mathematics to the intellectual level of daytime TV. So kick back, shift your brain into neutral, and we'll watch a few pictures whose only purpose is to leave you *relaxed* and confused."

I then had the great fortune to project onto a huge, bright screen, a live, relatively disaster free, forty minute color animation. (Let me assure those who attended that the backup videotape *would* have been a disaster by comparison. You just can't tape most of that stuff, or even convert it to current broadcast standard.)

Although the talk was fun, it contained little suitable for print publication. "The medium was the message." That is, the novelty was not so much in the mathematical concepts presented, but rather in the way that motion (particularly zooming) and color could vivify those concepts.

We look forward to an era of electronic publication in which widely available, muscular, high definition graphics engines will let us vivify for each other genuinely interesting mathematics.

Meanwhile, all I can offer is atonement for yet another misdeed at the aforementioned (NYU) conference—due to a two-year separation from TEX, I never contributed to the proceedings. Here then is a slightly less incoherent rendition of my 1984 talk.

1. Teaser

By strip mining, I meant applying the power of a modern symbolic processor, particularly via undetermined coefficients, to old fashioned investigations that were computationally inaccessible to the old fashioned investigators.

I feel sometimes like a kid at the controls of huge, bucket wheel excavator, chewing indiscriminately into apparently mined out formations, yet finding paydirt through sheer, childish brutality.

Random excavations, such as with Berlekamp's factoring algorithm, produce random nuggets:

$$\frac{(x-a)^3}{(z-x)^3(x-y)^3}+\frac{(y-a)^3}{(x-y)^3(y-z)^3}+\frac{(z-a)^3}{(y-z)^3(z-x)^3}=3\frac{(x-a)(y-a)(z-a)}{(x-y)^2(y-z)^2(z-x)^2}$$

$$
\begin{aligned}
\frac{1}{(w-x)^3(w-y)^3(w-z)^3} &+ \frac{1}{(x-y)^3(x-z)^3(x-w)^3} \\
&+ \frac{1}{(y-z)^3(y-w)^3(y-x)^3} + \frac{1}{(z-w)^3(z-x)^3(z-y)^3} \\
&= 3\,\frac{(w-x-y+z)(w-x+y-z)(w+x-y-z)}{(w-x)^2(w-y)^2(w-z)^2(x-y)^2(x-z)^2(y-z)^2},
\end{aligned}
$$

as does fooling with the trigonometric simplifiers:

$$
\begin{aligned}
&\sin(x-w)\sin(x-y)\sin(y-w)\sin(y-z)\sin(z-w)\sin(z-x) \\
+&\cos(x-w)\cos(x-y)\sin(y-w)\cos(y-z)\cos(z-w)\sin(z-x) \\
+&\cos(x-w)\sin(x-y)\cos(y-w)\cos(y-z)\sin(z-w)\cos(z-x) \\
+&\sin(x-w)\cos(x-y)\cos(y-w)\sin(y-z)\cos(z-w)\cos(z-x) = 0.
\end{aligned}
$$

These results are just like the factorization of x^2-y^2, or the sine addition formula, only millions of times less useful. They lead one to speculate whether algebraic engines in the Renaissance might have made this sort of thing into an art form.

More systematic excavations churn up bigger clumps of pay dirt:

$$
\begin{aligned}
&\sum_{n\geq 0}(1-q^{6n+\frac{1}{2}})\frac{(q^{-1/2},q^{3/2};q)_{2n}}{(q^2;q)_{4n}}\,\frac{(q^{5/2}/a,a;q^5)_n}{(q^{3/2}/a,a/q;q)_n}\,q^n \\
&= \frac{(\sqrt{q};q)_\infty(a;q^5)_\infty}{(q^2,a/q;q)_\infty}\left(\frac{(a/\sqrt{q};q)_\infty(q^5;q^5)_\infty}{(q^{5/2},aq^{5/2};q^5)_\infty} - \frac{a}{q}\,\frac{(\sqrt{q};q)_\infty(q^{5/2}/a;q^5)_\infty}{(q^{3/2}/a;q)_\infty}\sum_{n\geq 0}(-a^2)^n q^{5n^2/2}\right)
\end{aligned}
$$

is but one of dozens of identities that can be extracted from a single mathematical structure uncovered by a brute force attack on certain simultaneous polynomial equations. I call such a structure a "path invariant matrix system." Points in some space are linked by edges that are labeled with matrices in such a way that the matrix product taken along any path depends only on the endpoints. The path invariance condition on minimal loops induces polynomial equations among the matrix elements. A successful solution then provides a limitless space of closed contours, each yielding a correct identity, several of which may be concise enough to be interesting.

2. Matrices

Before getting into path invariance, let me extol some notational, analytic, and computational virtues of matrix products. First, notice how cumbersome are the conventional notations for an identity such as

$$\prod_{k\geq 1}\begin{pmatrix}\dfrac{k^2}{(4k+2)^2} & \dfrac{30k-11}{16k(2k-1)} \\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & \zeta(3) \\ 0 & 1\end{pmatrix}.$$

Hypergeometrically, you get a ridiculous $\frac{19}{16}{}_6F_5[\frac{1}{16}]$. As a straight sum, you can do as well as

$$\sum_{k\geq 1}\frac{30k-11}{4(2k-1)k^3\binom{2k}{k}^2} = \zeta(3),$$

but, taken literally, this says something rather silly: add up a sequence of terms whose kth contains products of k factors, in this case $\binom{2k}{k}^2$. Now most of us know the trick of incrementally computing each term from the previous, but this is nowhere hinted in the sum. Furthermore, you'll need a less-than-obvious auxiliary variable to avoid the wasted effort of cancelling out factors introduced in the term just preceding, or, more seriously, to avoid division by 0 if the numerator of the preceding term happens to vanish (*e.g*, if the 11 above were instead a 90).

Now suppose that you want a very precise value of $\zeta(3)$, and thus wish to sum n terms of this series. You will find it dramatically cheaper to write out instead the first n matrices, and then pairwise multiply to form $\lceil n/2\rceil$ products, and repeat until only one matrix remains. (In 1985, I used this technique, essentially due to R. Schroeppel, to temporarily steal the π computation record from Japan.)

This representation of simple sums as 2×2 matrix products is merely a special case (a vanishing off-diagonal) of the well known representation of nested homographic functions. When a diagonal element vanishes, you have a continued fraction. It is strange that so fruitful a representation is applied so rarely to sums. Perhaps discouraging is the apparent nonlinearity of the matrix product form, (partially) concealing such familiar operations as termwise differentiation or combination with other series. We shall see that a small bag of tricks, again based on path invariance, more than remedies these problems.

Even greater simplifications are possible with larger matrices:

$$\sum_{n=0}^{m} a_n x^{n-1}\sum_{k=0}^{n-1}\frac{b_k}{k!} = \prod_{n=0}^{m}\begin{pmatrix}\dfrac{x}{n+1} & b_n & 0 \\ 0 & x & a_n \\ 0 & 0 & 1\end{pmatrix}_{1,3},$$

where the subscript$_{1,3}$ means the upper right element. Here, the matrix product not only vaporizes a nested loop, but also dispels an artificial asymmetry between a_i and b_i. They can, in fact, be switched via summation by parts, but then they are asymmetrical the other way!

And here is a way to get factorable multiple sums:

$$\sum_{i=0}^{m}\sum_{j=0}^{m} a_i b_j = \prod_{n=0}^{m} \begin{pmatrix} 1 & a_n & b_n & a_n b_n \\ 0 & 1 & 0 & b_n \\ 0 & 0 & 1 & a_n \\ 0 & 0 & 0 & 1 \end{pmatrix}_{1,4} .$$

Perhaps most important of all are those sadly neglected recurrences that can be expressed by products of denser, (typically) nontriangular matrices. Who knows what we might find, once emancipated from our feebly notated sums, (scalar) products, and continued fractions? One enticement is a smooth generalization of the notion of "closed form": a recurrence is in simpler (and therefore more canonical) form when it is the product of smaller or sparser matrices. Fully closed form is a 1×1, *i.e.* scalar product. The importance of such products, by the way, is almost purely their uniqueness and comparability, since they usually converge more slowly than most of the higher order recurrences that they canonicalize.

3. Path Invariance

This idea was crystalized by Kevin Karplus while a student in the only course I ever taught. Label the edges of some sort of directed, multiply connected graph with matrices so that their product, taken along any connected sequence of edges, depends only on the endpoints. Usually, the graph is piecewise grid-like, and it is sufficient to demonstrate path invariance around each of the grid cells, and around the (usually) triangular cells where the grids patch together. For example, a purely two dimensional, path invariant grid would look like

$k \uparrow$

$\rightarrow$
n

$$\begin{tikzcd}
 & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \\
\cdots \arrow[r] & \bullet \arrow[rr, "N_{k+1,n}"] & & \bullet \arrow[r, dash] & \cdots \\
 & & & & \\
\cdots \arrow[r] & \bullet \arrow[uu, "K_{k,n}"] \arrow[rr, "N_{k,n}"] & & \bullet \arrow[uu, "K_{k,n+1}"'] \arrow[r, dash] & \cdots \\
 & \vdots \arrow[u] & & \vdots \arrow[u] &
\end{tikzcd}$$

where the matrices $N_{k,n}$ and $K_{k,n}$ satisfy the invariance

$$N_{k,n}K_{k,n+1} = K_{k,n}N_{k+1,n},$$

a sort of discrete Cauchy-Riemann condition. For example, we might have

$$N_{k,n} := \begin{pmatrix} \dfrac{q^k}{1-q^{n+1}} & 1-q^k \\ 0 & 1 \end{pmatrix}, \qquad K_{k,n} := \begin{pmatrix} \dfrac{q^n}{1-q^{k+1}} & 1-q^n \\ 0 & 1 \end{pmatrix}.$$

Then

$$N_{k,n}K_{k,n+1} = K_{k,n}N_{k+1,n} = \begin{pmatrix} \dfrac{q^{n+k+1}}{(1-q^{n+1})(1-q^{k+1})} & 1 \\ 0 & 1 \end{pmatrix}.$$

If we now equate the two path products along a rectangle starting at $n = 0, k = a$ and closing at $n = n_{\max}, k = k_{\max}$, and then let $(k_{\max}, n_{\max}) \to (\infty, \infty)$ (however they wish, in this case), then we have, after a limit interchange justifiable when $|q| < 1$,

$$\prod_{n\geq 0} \begin{pmatrix} \dfrac{q^a}{1-q^{n+1}} & 1-q^a \\ 0 & 1 \end{pmatrix} \prod_{k\geq a} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \prod_{k\geq a} \begin{pmatrix} \dfrac{1}{1-q^{k+1}} & 0 \\ 0 & 1 \end{pmatrix} \prod_{n\geq 0} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Expanding the infinite products, (and, for convergence, assuming $|q^a| < 1$, *i.e.* $\Re a > 0$),

$$\begin{pmatrix} 0 & (1-q^a)\left(1+\frac{q^a}{1-q}\left(1+\frac{q^a}{1-q^2}\left(1+\cdots\right)\right)\right) \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{1-q^{a+1}}\,\dfrac{1}{1-q^{a+2}}\,\dfrac{1}{1-q^{a+3}}\cdots & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

or, equating the upper right elements and dividing by $1-q^a$,

$$1+\frac{q^a}{1-q}+\frac{q^{2a}}{(1-q)(1-q^2)}+\frac{q^{3a}}{(1-q)(1-q^2)(1-q^3)}+\cdots = \frac{1}{1-q^a}\,\frac{1}{1-q^{a+1}}\,\frac{1}{1-q^{a+2}}\cdots.$$

Given path invariance, we can adjoin the reverses of those edges with invertible labels. And we can freely shortcut any connected sequence of edges by a single edge, since its associated matrix is uniquely determined. In particular, we can draw the diagonals J_j connecting those nodes where $k - n = a$:

$$
\begin{tikzcd}
 & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & J_{j+1} \\
\cdots \arrow[r] & \bullet \arrow[rr, "N_{j+a+1,j}"] & & \bullet \arrow[r, dash] & \cdots \\
\cdots \arrow[r] & \bullet \arrow[u, "K_{j+a,j}"] \arrow[rr, "N_{j+a,j}"] \arrow[urr, phantom, "J_j"] & & \bullet \arrow[u, "K_{j+a,j+1}"'] \arrow[r, dash] & \cdots \\
J_{j-1} & \vdots \arrow[u] & & \vdots \arrow[u] &
\end{tikzcd}
$$

Then

$$J_j := N_{j+a,j} K_{j+a,j+1} = K_{j+a,j} N_{j+a+1,j} = \begin{pmatrix} \dfrac{q^{2j+a+1}}{(1-q^{j+1})(1-q^{j+a+1})} & 1 \\ 0 & 1 \end{pmatrix},$$

and the shortcut across the whole rectangle is equivalent to the previous edge traversals:

$$\prod_{j \geq 0} J_j = \prod_{n \geq 0} N_{a,n} \prod_{k \geq a} K_{k,\infty} = \prod_{k \geq a} K_{k,0} \prod_{n \geq 0} N_{\infty,n},$$

that is

$$
\begin{aligned}
1 + & \frac{q^{a+1}}{(1-q)(1-q^{a+1})} + \frac{q^{2a+4}}{(1-q)(1-q^2)(1-q^{a+1})(1-q^{a+2})} \\
& + \frac{q^{3a+9}}{(1-q)(1-q^2)(1-q^3)(1-q^{a+1})(1-q^{a+2})(1-q^{a+3})} + \cdots \\
& = (1-q^a)\Big(1 + \frac{q^a}{1-q} + \frac{q^{2a}}{(1-q)(1-q^2)} + \frac{q^{3a}}{(1-q)(1-q^2)(1-q^3)} + \cdots\Big) \\
& = \frac{1}{1-q^{a+1}} \frac{1}{1-q^{a+2}} \frac{1}{1-q^{a+3}} \cdots .
\end{aligned}
$$

The quadratically progressing exponent in the diagonal sum enhances convergence both numerically, and in the symbolic power series expansion at $q = 0$.

4. One step transformations

The grids for most of the standard series transformations (such as Kummer's or Euler's) are just "railroad tracks". One rail (call it the $j = 0$ rail) is labeled with the original matrix sequence $N_{0,n}$, and the "ties", labeled with matrices J_n, mechanize the transformation to the $j = 1$ rail. This same geometry mechanizes the termwise transformation of a composition of arbitrary homographic functions (possibly a sum!) into a (typically, non-regular) continued fraction. That is, let

$$N_{0,n} := \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

be the matrix for the homographic function

$$f_n(x) := \frac{a_n x + b_n}{c_n x + d_n},$$

and define

$$g_n := a_n + \frac{c_n}{c_{n-1}} d_{n-1}, \quad h_n := -\frac{c_{n+1}}{c_n}(a_n d_n - b_n c_n).$$

Here, we must have $c_n \neq 0$, so any sum must be via $b_n = 0$. Then the "tie" matrices are

$$J_n := \begin{pmatrix} 1 & a_n - g_n \\ 0 & c_n \end{pmatrix}$$

and the continued fraction ($j = 1$) rail has matrices

$$N_{1,n} := \begin{pmatrix} g_n & h_n \\ 1 & 0 \end{pmatrix}.$$

Path invariance follows from mechanically verifying $N_{0,n} J_{n+1} = J_n N_{1,n}$. Then the equivalence of the paths $(0,1) \to (0,m) \to (1,m)$ and $(0,1) \to (1,1) \to (1,m)$, after multiplying on the right by the columnating equation defining x,

$$\begin{pmatrix} \left(y + \frac{d_m}{d_{m+1}}\right) c_{m+1} \\ 1 \end{pmatrix} =: \begin{pmatrix} x \\ 1 \end{pmatrix},$$

and equating the ratios of the upper and lower elements, gives the desired identity:

$$f_1(f_2(\dots(f_m(y))\dots)) = -\frac{d_0}{c_0} + \frac{1}{c_1}\Bigg(g_1 + \cfrac{h_1}{g_2 + \cfrac{h_2}{\ddots \, g_m + \cfrac{h_m}{x}}}\Bigg).$$

To convert a sum from the standard form ($c_n = 0$), adjoin it as a $N_{-1,n}$ "third rail" via ties $J'_n := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, or any other matrix which unzeros the lower left element of $N_{0,n} = J_n'^{-1} N_{-1,n} J'_{n+1}$.

Even the artifice of multiplication by $\binom{x}{1}$ fits into the path invariance scheme, merely by connecting a new, "black hole" node to all existing nodes via edges labeled with this (uninvertible) matrix.

Another transformation, summation by parts, is applicable either before or after conversion to ratio form, via a "double decker" railroad track in the form of a square tube. Let p_n and q_n be two arbitrary functions of the summation index n. Traditional summation by parts transpires on the bottom ($j = 0$) track, via the matrices

$$N_{0,0,n} := \begin{pmatrix} 1 & q_n(p_{n+1} - p_n) \\ 0 & 1 \end{pmatrix}, \qquad N_{0,1,n} = \begin{pmatrix} 1 & p_{n+1}(q_{n+1} - q_n) \\ 0 & 1 \end{pmatrix}$$

$$K_{0,0,n} := \begin{pmatrix} -1 & -p_n q_n \\ 0 & 1 \end{pmatrix},$$

where the subscript order is j, k, n. Now suppose that

$$\rho_n := \frac{p_{n+1}}{p_n}, \quad \text{and} \quad \sigma_n := \frac{q_{n+1}}{q_n}$$

are "nicer" functions of n than the corresponding p_n and q_n, perhaps by virtue of factorials or nth powers in the latter. Then one can jump to the upper track, where summation by parts looks like

$$N_{1,0,n} = \begin{pmatrix} \rho_n\sigma_n & \rho_n - 1 \\ 0 & 1 \end{pmatrix}, \qquad N_{1,1,n} = \begin{pmatrix} \rho_n\sigma_n & \rho_n(\sigma_n - 1) \\ 0 & 1 \end{pmatrix}$$

$$K_{1,0,n} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

if one labels the "vertical ties" with

$$J_{0,k,n} := \begin{pmatrix} p_n q_n & 0 \\ 0 & 1 \end{pmatrix}.$$

For example, with

$$\rho_n := \frac{n+c}{n+e}\,\frac{n+d}{n+c+d-e}, \qquad \sigma_n := \frac{n+a+1}{n+1}\,\frac{n+b+1}{n+a+b+1},$$

the path equivalence $(1,0,0) \to (1,1,0) \to (1,1,\infty) = (1,0,0) \to (1,0,\infty) \to (1,1,\infty)$, *i.e.*

$$K_{1,0,0} \prod_{n\geq 0} N_{1,1,n} = \left(\prod_{n\geq 0} N_{1,0,n} \right) K_{1,0,\infty},$$

gives, after negating the upper right elements,

$$\begin{aligned} {}_4F_3\left[\begin{matrix} a, & b, & c, & d \\ & a+b+1, & e, & c+d-e \end{matrix}\right] &= \frac{(a+b)!\,(c+d-e-1)!\,(e-1)!}{a!\,b!\,(c-1)!\,(d-1)!} \\ &\quad + \frac{e-c}{e}\,\frac{e-d}{e-c-d}\,{}_4F_3\left[\begin{matrix} a+1, & b+1, & c, & d \\ & a+b+1, & e+1, & c+d-e+1 \end{matrix}\right]. \end{aligned}$$

When only one of σ_n and ρ_n is nice, it is straightforward to construct diagonal ties linking an upper rail with a lower one, getting a hybrid, partially rationalized summation by parts.

5. Telescopy

Telescopy takes place between a single rail and a black hole. The problem is to find columnar, "diving in" matrices which path-invariantly connect the black hole to all the nodes along the rail. Then the telescoping identity is the equivalence between diving straight in, and performing the sum prior to diving in.

$$\begin{pmatrix} 1 & b_n - b_{n-1} \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -b_n \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -b_{n+1} \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & b_{n+2} - b_{n+1} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & b_{n+1} - b_n \\ 0 & 1 \end{pmatrix}$$

A nontrivial choice of b_n might be

$$b_n := \frac{2^{-n}}{1 - x^{2^{-n}}} \quad \Rightarrow \quad \sum_{n \geq 1} \frac{2^{-n}}{1 + x^{2^{-n}}} = \frac{1}{\log x} + \frac{1}{1 - x},$$

when the alternate paths dive in from $n = 1$ and $n = \infty$.

Rationalized telescopy differs from regular telescopy only in the contents of the matrices.

$$\begin{pmatrix} \dfrac{1 + f_n}{f_{n+1}} & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -f_{n+1} \\ 1 \end{pmatrix} = \begin{pmatrix} -f_n \\ 1 \end{pmatrix} \Rightarrow \prod_{n=0}^{m} \begin{pmatrix} \dfrac{1 + f_n}{f_{n+1}} & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -f_{m+1} \\ 1 \end{pmatrix} = \begin{pmatrix} -f_0 \\ 1 \end{pmatrix}$$

An interesting choice of f_n is

$$f_n := -\frac{x^{2^{-n}} + c}{c^2 + c} \quad \Rightarrow \quad \sum_{n \geq 0} (\sqrt{x} - c)(\sqrt{\sqrt{x}} - c) \ldots (x^{2^{-n}} - c) = \frac{\frac{x}{c} + 1}{c + 1}, \qquad |1 - c| < 1.$$

Any hypergeometric sum, that is, one whose term ratio is a rational function of the summation index n, can be expressed in numerous ways as a product of matrices of polynomials in n. We now have infallible algorithms (decision procedures) for determining, where possible, arbitrary partial sums of such series. At least one of these algorithms [1] hinges on minimizing the degrees of the on-diagonal polynomials at the expense of the non-zero off-diagonal one. This canonicalization proceeds in a finite number of steps, and can be visualized as path invariant hopping between parallel rails to the minimal one. The minimality then reduces the search for the telescoper ("diving in") matrix to a single system of simultaneous linear equations.

6. "Residues", Euler-Maclaurin summation

Let f be sufficiently tame, and define

$$N_{k,n} := \begin{pmatrix} 1 & \int_0^1 f^{(k)}(t+n) B_k(t)\, dt \\ 0 & 1 \end{pmatrix}, \quad K_{k,n} := \begin{pmatrix} -\dfrac{1}{k+1} & -\dfrac{B_{k+1}}{k+1} f^{(k)}(n) \\ 0 & 1 \end{pmatrix}.$$

Then path invariance,

$$N_{k,n} K_{k,n+1} = K_{k,n} N_{k+1,n}, \qquad k > 0$$

follows from integration by parts, except when $k = 0$, where

$$N_{0,n} K_{0,n+1} = E_n K_{0,n} N_{1,n},$$

$$E_n := \begin{pmatrix} 1 & f_{n+1} \\ 0 & 1 \end{pmatrix} = N_{0,n} K_{0,n+1} N_{1,n}^{-1} K_{0,n}^{-1},$$

because 1 is the only non-negative integer k for which $B_k(0) \neq B_k(1)$. Thus, path invariance breaks down between $k = 0$ and $k = 1$, and the residues are tallied by the E_n, which commute to the left. In the final formula, these residues comprise the $\sum f(n)$ to which the $\int f(t)$ (from the $k = 0$ path) is the first approximation.

$$\begin{aligned}
&\prod_{n=a}^{b} N_{0,n} \prod_{k=0}^{c} K_{k,b+1} = \prod_{j=a}^{b} E_j \prod_{k=0}^{c} K_{k,a} \prod_{n=a}^{b} N_{c+1,n} \\
&\quad \Leftrightarrow \quad \int_a^{b+1} f(t)\, dt + \sum_{k=0}^{c} \frac{(-)^{k+1}}{(k+1)!} B_{k+1} f^{(k)}(b+1) \\
&\quad = \sum_{n=a}^{b} f(n+1) + \sum_{k=0}^{c} \frac{(-)^{k+1}}{(k+1)!} B_{k+1} f^{(k)}(a) + \frac{(-)^{c+1}}{(c+1)!} \sum_{n=a}^{b} \int_0^1 f^{(c+1)}(t+n) B_{c+1}(t)\, dt
\end{aligned}$$

7. ${}_3F_2[1]$ Rosetta stone

$$N_{g,h,i,j,k,n} := \begin{pmatrix} \frac{n+h}{n+1}\frac{n+i}{n+g}\frac{n+j}{n+k} & 0 & n \\ 0 & \frac{n+h}{n+1}\frac{n+i}{n+g}\frac{n+j}{n+k} & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$K_{g,h,i,j,k,n} :=$

$$\begin{pmatrix} \frac{(k-h)(k-i)(k-j)+hij}{k(k+g-h-i-j-1)(k+n)} & \frac{hij}{(k+g-h-i-j-1)(k+n)} & \frac{n(n+g-1)}{k+g-h-i-j-1} \\ \frac{1}{k+n} & \frac{k}{k+n} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$J_{g,h,i,j,k,n} :=$

$$\begin{pmatrix} \frac{(j-k-g+h+i+2)(j+n)}{(j-g+1)(j-k+1)} & \frac{hi(j+n)}{(j-g+1)(j-k+1)} & n\frac{n+g-1}{j-g+1}\frac{n+k-1}{j-k+1} \\ -\frac{(j-k-g+h+i+2)(j+n)}{j(j-g+1)(j-k+1)} & \frac{j+n}{j}\left(1-\frac{h}{j-g+1}\frac{i}{j-k+1}\right) & -\frac{n}{j}\frac{n+g-1}{j-g+1}\frac{n+k-1}{j-k+1} \\ 0 & 0 & 1 \end{pmatrix}$$

This six-dimensional path invariant grid codifies all of the ${}_3F_2[1]$ contiguity relations (and, as a limiting case, all the ${}_2F_1[z]$ and ${}_1F_1[z]$). The three undisplayed matrices follow from symmetry: $G_{g,h,i,j,k,n} = K_{k,h,i,j,g,n}$, etc. The ${}_3F_2[1]$ arises in element$_{2,3}$ of $\prod_{n\geq 0} N_{\dots,n}$. Any contiguous sum can be reached in a finite number of multiplications by the parameter gunching matrices. The contiguity identity follows by closing the contour at $n=\infty$.

Contours that range indefinitely in indices other than n yield identities involving the general ${}_3F_2[1]$ which are unsurprisingly resistant to conventional notation, due to the non-triangularity of the non-N matrices. But even if we never become facile with them, at least some such identities should be valuable for their rapid numerical convergence.

Perhaps more interestingly, by conceding linear constraints among the parameter indices, we can triangularize some of their matrices, and get all of the familiar ${}_3F_2[1]$ identities (Dixon's, Saalschütz's, Whipple's, Watson's, etc.). When you change coordinates, Whipple's becomes Watson's, and Saalschütz's becomes the very well poised ${}_5F_4$.

The imposition of a linear constraint entails the replacement of several indices by a new one. This corresponds to recoordinatization and then dimension reduction in the grid. An example is given in section **9** ("**An application**").

As often happens when a system like the above is symmetric in several indices, one can find a corresponding system with those symmetric indices collapsed into one, and running at fractional speed, *e.g.* with (g,k) replaced by $(\frac{k}{2},\frac{k+1}{2})$, or (h,i,j) replaced by $(\frac{j}{3},\frac{j+1}{3},\frac{j+2}{3})$. Another way to put it is that, in the (g,k) case, the matrix $G_{k,h,i,j,k,n}K_{k+1,h,i,j,k,n}$ can be factored into the form $M(2k)M(2k+1)$. Such parameter specializations often simplify the quest for triangularizations, and lead to interesting identities upon subsequent coordinate changes.

Letting parameters, say h and g, blow up in a fixed relative ratio z yields a four-dimensional system for ${}_2F_1[z]$, which is easier to strip mine for triangularizations, particularly when z is negotiable. Further specializing via the abovementioned "fractional parameter collapse" led me to

$$
{}_2F_1\left[\begin{matrix}-\frac{a}{2}, & \frac{1-a}{2} \\ & 2a+\frac{3}{2}\end{matrix}\middle|\,\frac{1}{5}\right] = \sqrt{\pi\frac{5+\sqrt{5}}{5}}\;\frac{2^{5a+\frac{3}{2}}}{5^{3a+1}}\;\frac{(2a+\frac{1}{2})!}{(a-\frac{1}{5})!(a+\frac{1}{5})!}
$$

$$
{}_2F_1\left[\begin{matrix}-\frac{a}{2}, & \frac{1-a}{2} \\ & 2a+\frac{5}{2}\end{matrix}\middle|\,\frac{1}{5}\right] = \sqrt{\pi\frac{5-\sqrt{5}}{5}}\;\frac{2^{5a+\frac{7}{2}}}{5^{3a+2}}\;\frac{(2a+\frac{3}{2})!}{(a+\frac{2}{5})!(a+\frac{3}{5})!}.
$$

while strip mining the remaining degrees of freedom. (Professor P. Karlsson has lately informed me that these were dug up by W. Heymann in 1898 ([2],[3]), *i.e.*, with his bare hands!)

I also have q-versions of the ${}_3F_2[1]$ system. These have the annoying property of sometimes leaving factors of $1-q$ instead of 0 when subjected to the transformations which triangularize their $q=1$ cousins, which is one way to explain why we yet lack q-generalizations for some of our more exotic ${}_2F_1$ identities.

8. Continued fractions

This is three N-K planes, $j=-1,0,1$, joined by two J matrices. In the top ($j=1$) plane, the k direction computes a continued fraction, and on the bottom ($j=-1$), it's the n direction.

$$
N_{1,k,n} = \begin{pmatrix}(1-n-k)z & a\big((k-1)^2+b\big)z^2 \\ 1 & a(n-k+1)z\end{pmatrix},
$$

$$
K_{1,k,n} = \begin{pmatrix}\big((a-1)k-(a+1)n+1\big)z & a(k^2+b)z^2 \\ 1 & 0\end{pmatrix}
$$

$$
J_{0,k,n} := \begin{pmatrix}z^{n+k} & \big((a+1)n+k-1\big)z^{n+k+1} \\ 0 & z^{n+k+1}\end{pmatrix}
$$

$$N_{0,k,n} := \begin{pmatrix} an & a\big(b-(a+1)kn\big) \\ 1 & -n-(a+1)k \end{pmatrix}, \quad K_{0,k,n} := \begin{pmatrix} ak & a\big(b-(a+1)nk\big) \\ 1 & -k-(a+1)n \end{pmatrix}$$

$$J_{-1,k,n} := \begin{pmatrix} z^{-n-k} & \big(1-(a+1)k-n\big)z^{-n-k} \\ 0 & z^{-n-k-1} \end{pmatrix}$$

$$N_{-1,k,n} = \begin{pmatrix} \big((a-1)n-(a+1)k+1\big)z & a(n^2+b)z^2 \\ 1 & 0 \end{pmatrix},$$

$$K_{-1,k,n} = \begin{pmatrix} (1-k-n)z & a\big((n-1)^2+b\big)z^2 \\ 1 & a(k-n+1)z \end{pmatrix}$$

Thus, to get a relation between two continued fractions, we must j-hop between planes before switching directions. If k runs from c to ∞ and n runs from d to ∞, and the corresponding continued fractions converge, then they are insensitive to whatever subsequent matrices accrue on the right by way of path closure. *I.e.*, the two paths can be $(j,k,n) = (0,c,d) \to (1,c,d) \to (1,\infty,d)$ and $(0,c,d) \to (-1,c,d) \to (-1,c,\infty)$. This yields the identity

$$f(c,d) = f(d,c)$$

where

$$f(x,y) := ((a+1)y+x-1)\,z \\ + ((a-1)x-(a+1)y+1)\,z + \cfrac{a(x^2+b)z^2}{((a-1)(x+1)-(a+1)y+1)\,z + \cfrac{a\left((x+1)^2+b\right)z^2}{\ddots}}.$$

Note that z can be simply canceled out of this identity, but it is handy to leave it in. Since we never travel in the $j=0$ plane, you might consider discarding it and coalescing the two J matrices. My only objections to this are that it would destroy some symmetry, and more importantly, it would conceal the simple derivation, which just uses the technique in section **4** to force alternately the K and N matrices into continued fraction form.

Another application of this particular path invariant system is a direct interderivation of the series and continued fraction forms for arctan, in the form

$$1+\cfrac{x^2}{3+\cfrac{4x^2}{5+\cfrac{9x^2}{\ddots}}} = \frac{x}{\arctan x} = \frac{\frac{2y}{1-y^2}}{2\arctan y}$$

$$= \frac{1}{(1-y^2)\left(1-\frac{y^2}{3}+\frac{y^4}{5}-\cdots\right)},$$

where

$$x := \frac{2y}{1-y^2}.$$

We merely remain in the $j=-1$ plane and get our series by choosing the parameters to annihilate the upper right element of K:

$$a := y, \qquad b := 0, \qquad z := -\frac{2}{1-y^2}.$$

This gives

$$F_n := N_{-1,\frac{1}{2},n} = \begin{pmatrix} 2n-1 & n^2\left(\frac{2y}{1-y^2}\right)^2 \\ 1 & 0 \end{pmatrix}$$

and

$$G_k := K_{-1,k,1} = \begin{pmatrix} \frac{2k}{1-y^2} & 0 \\ 1 & -\frac{2ky^2}{1-y^2} \end{pmatrix}.$$

Then the ratio of the lefthand elements of $F_1F_2F_3\ldots$ gives the continued fraction, while the corresponding ratio from $G_{\frac{1}{2}}G_{\frac{3}{2}}G_{\frac{5}{2}}\ldots$ gives the reciprocated series.

9. An application

$$J_{j,k,n} := \begin{pmatrix} \frac{1-q^{j-k+n+\frac{1}{2}}}{1-q^{2j+2}} \frac{1-q^{j+k-n+\frac{1}{2}}}{1-q^{j-k-n+\frac{1}{2}}} & \frac{1-q^{2k}}{1-q} \frac{1-q^{2n}}{1-q^{j-k-n+\frac{1}{2}}} q^{j-k-n+\frac{1}{2}} \\ 0 & 1 \end{pmatrix}$$

$$K_{j,k,n} := \begin{pmatrix} \frac{1-q^{k-n+j+\frac{1}{2}}}{1-q^{2k+2}} \frac{1-q^{k+n-j+\frac{1}{2}}}{1-q^{k-n-j+\frac{1}{2}}} & \frac{1-q^{2n}}{1-q} \frac{1-q^{2j}}{1-q^{k-n-j+\frac{1}{2}}} q^{k-n-j+\frac{1}{2}} \\ 0 & 1 \end{pmatrix}$$

$$N_{j,k,n} := \begin{pmatrix} \frac{1-q^{n-j+k+\frac{1}{2}}}{1-q^{2n+2}} \frac{1-q^{n+j-k+\frac{1}{2}}}{1-q^{n-j-k+\frac{1}{2}}} & \frac{1-q^{2j}}{1-q} \frac{1-q^{2k}}{1-q^{n-j-k+\frac{1}{2}}} q^{n-j-k+\frac{1}{2}} \\ 0 & 1 \end{pmatrix}$$

This pretty, three-dimensional system, when path multiplied around an infinite rectangle based at $j = a, k = b, n = 0$ in either the k-n or n-j plane, gives Andrews's q-gneralization of Bailey's ${}_2F_1[\frac{1}{2}]$ theorem ([4]). (And, by symmetry, there are four other ways to do it.)

By making the coordinate transformation $n \leftarrow n + j + k + \frac{1}{2}$, you get Andrews's q-gneralization of Gauss's ${}_2F_1[\frac{1}{2}]$ (also [4]). To preserve path invariance, this coordinate change requires that the $J_{j,k,n}$ and $K_{j,k,n}$ matrices be replaced by $J_{j,k,n}N_{j+1,k,n}$ (or $N_{j,k,n}J_{j,k,n+1}$) and $K_{j,k,n}N_{j,k+1,n}$ (or $N_{j,k,n}K_{j,k,n+1}$) respectively, followed by the actual substitution $n \leftarrow n + j + k + \frac{1}{2}$. The multiplication of J and K by N reflects that every incrementation of j or k must also increment what was formerly n.

A more interesting coordinate change, in greater detail, is

$$\left\{ \begin{matrix} J_{j,k,n} \\ K_{j,k,n} \\ N_{j,k,n} \end{matrix} \right\} \leftarrow \left\{ \begin{matrix} J_{j,k+j,n+k+j}\, K_{j+1,k+j,n+k+j}\, N_{j+1,k+j+1,n+k+j} \\ K_{j,k+j,n+k+j}\, N_{j,k+j+1,n+k+j} \\ N_{j,k+j,n+k+j} \end{matrix} \right\}$$

which gives

$$J_{j,k,n} = \begin{pmatrix} -\frac{1-q^{j-n+\frac{1}{2}}}{1-q^{2(j+1)}}\frac{1-q^{j+n+\frac{1}{2}}}{1-q^{2(j+k+1)}}\frac{1-q^{j+2k+n+\frac{1}{2}}}{1-q^{2(j+k+n+1)}}q^{3j+2k+n+\frac{3}{2}} & -\frac{1-q^{3j+2k+n+\frac{1}{2}}}{1-q} \\ 0 & 1 \end{pmatrix}$$

$$K_{j,k,n} = \begin{pmatrix} \frac{1-q^{2k+j+n+\frac{1}{2}}}{1-q^{2(k+j+1)}}\frac{1-q^{2k+j+n+\frac{3}{2}}}{1-q^{2(k+j+n+1)}}q^{2j} & -\frac{1-q^{2j}}{1-q} \\ 0 & 1 \end{pmatrix}$$

$$N_{j,k,n} = \begin{pmatrix} \frac{1-q^{n+j+\frac{1}{2}}}{1-q^{n-j+\frac{1}{2}}}\frac{1-q^{n+j+2k+\frac{1}{2}}}{1-q^{2(n+j+k+1)}} & \frac{1-q^{2j}}{1-q}\frac{1-q^{2(j+k)}}{1-q^{n-j+\frac{1}{2}}}q^{n-j+\frac{1}{2}} \\ 0 & 1 \end{pmatrix}.$$

Then a rectangle in the j-k plane, based at $j=0, k=(b-a)/2, n=a-\frac{1}{2}$, gives

$$\prod_{j\geq 0}\begin{pmatrix} -\frac{1-q^{j+a}}{1-q^{2j+2}}\frac{1-q^{j+1-a}}{1-q^{2j+b-a+2}}\frac{1-q^{j+b}}{1-q^{2j+b+a+1}}q^{3j+b+1} & \frac{1-q^{3j+b}}{1-q} \\ 0 & 1 \end{pmatrix}_{1,2}$$

$$= \frac{\frac{b-a}{2}!_{q^2}}{(-\frac{1}{2})!_{q^2}}\frac{\frac{b+a-1}{2}!_{q^2}}{(b-1)!_q}(1+q)^b,$$

where

$$z!_q := (1-q)^{-z}\prod_{n\geq 1}\frac{1-q^n}{1-q^{n+z}} \text{ “ = ” } \frac{1-q}{1-q}\frac{1-q^2}{1-q}\cdots\frac{1-q^z}{1-q}.$$

Alternatively, writing $q^a =: A, q^b =: B$, and multiplying through by $1-q$,

$$\sum_{j\geq 0}(1-Bq^{3j})\frac{(A,q/A,B;q)_j}{(q^2,ABq,Bq^2/A;q^2)_j}(-Bq^{(3j-1)/2})^j=\frac{(B;q)_\infty}{(ABq,Bq^2/A;q^2)_\infty}.$$

Letting $q\to 1$ in the penultimate equation, and dividing through by b,

$${}_4F_3\left[\begin{matrix}a, & 1-a, & b, & \frac{b}{3}+1\\ & \frac{b-a}{2}+1, & \frac{b+a+1}{2}, & \frac{b}{3}\end{matrix}\middle| -\frac{1}{8}\right]=\frac{\frac{b-a}{2}!}{(-\frac{1}{2})!}\frac{\frac{b+a-1}{2}!}{b!}2^b$$

This gives a rapidly convergent (three bits per term) series for the useful but very slowly (if at all) convergent ${}_2F_1[a,b;c+1|1]$:

$$\frac{c!\,(c-a-b)!}{(c-a)!\,(c-b)!}={}_2F_1\left[\begin{matrix}a, & b\\ & c+1\end{matrix}\middle| 1\right]$$

$$=\frac{{}_4F_3\left[\begin{matrix}\frac{1}{2}+a+b, & \frac{1}{2}-a-b, & 2c-a-b+\frac{1}{2}, & \frac{2c-a-b}{3}+\frac{7}{6}\\ & c+1, & c-a-b+1, & \frac{2c-a-b}{3}+\frac{1}{6}\end{matrix}\middle| -\frac{1}{8}\right]}{{}_4F_3\left[\begin{matrix}a-b+\frac{1}{2}, & b-a+\frac{1}{2}, & 2c-a-b+\frac{1}{2}, & \frac{2c-a-b}{3}+\frac{7}{6}\\ & c-a+1, & c-b+1, & \frac{2c-a-b}{3}+\frac{1}{6}\end{matrix}\middle| -\frac{1}{8}\right]}$$

$$=\frac{{}_4F_3\left[\begin{matrix}c-2b-a+1, & -c+2b+a, & c-a, & \frac{c-a}{3}+1\\ & c-b-a+1, & b+\frac{1}{2}, & \frac{c-a}{3}\end{matrix}\middle| -\frac{1}{8}\right]}{{}_4F_3\left[\begin{matrix}c-2b+1, & -c+2b, & c, & \frac{c}{3}+1\\ & c-b+1, & b+\frac{1}{2}, & \frac{c}{3}\end{matrix}\middle| -\frac{1}{8}\right]}2^a$$

$$=\frac{{}_4F_3\left[\begin{matrix}c+a+1, & -c-a, & c-a, & \frac{c-a}{3}+1\\ & c+1, & \frac{1}{2}-a, & \frac{c-a}{3}\end{matrix}\middle| -\frac{1}{8}\right]}{{}_4F_3\left[\begin{matrix}c-b+a+1, & -c+b-a, & c-b-a, & \frac{c-b-a}{3}+1\\ & c-b+1, & \frac{1}{2}-a, & \frac{c-b-a}{3}\end{matrix}\middle| -\frac{1}{8}\right]}2^{-b}.$$

Even if you neglect to use matrix products to evaluate the ${}_4F_3$s, these are notably cheaper than four invocations of the Γ function, especially when a,b, and c might be complex. Also, the (complex) Beta function is the special case

$$\mathrm{B}(a,b)=\frac{{}_2F_1[a-1,-b;a|1]}{b}=\frac{1}{(a+b-1)\,{}_2F_1[1-a,1-b;1|1]}=\frac{\pi\sin\pi(a+b)}{\sin\pi a\,\sin\pi b}{}_2F_1\left[\begin{matrix}a, & b\\ & 1\end{matrix}\middle| 1\right].$$

10. A commercial for q-trigonometry

In the preceding section, we saw both the factorial and q^2-factorial of $-1/2$. The former is $\sqrt{\pi}$; why not call the latter $\sqrt{\pi}_q$? More generally, why don't we have a q-version of the factorial reflection formula $z!\,(-z)! = \pi z/\sin \pi z$? I propose

$$\begin{aligned}
\Pi_q := \frac{\pi_q}{1-q^2} &= q^{\frac14}\frac{(-\frac12)!_{q^2}^2}{1-q^2} = q^{\frac14}\prod_{n\ge1}\frac{(1-q^{2n})^2}{(1-q^{2n-1})^2} = \frac{q\text{-Wallis product}}{1-q}\\
&= q^{\frac14}(1+q+q^3+q^6+q^{10}+\cdots)^2 = \frac{\vartheta_2\vartheta_3}{2} = \frac{\vartheta_2^2(0,q^{\frac12})}{4}\\
&= \lim_{z\to0}\frac{\sin_q \pi z}{q^{-z}-q^z} = -\frac{\pi}{2}\frac{\sin_q' 0}{\ln q} = \frac{\eta(q^2)^4}{\eta(q)^2}
\end{aligned}$$

where

$$\begin{aligned}
\sin_q \pi z := \frac{q^{z(z-1)}\pi_q}{(z-1)!_{q^2}\,(-z)!_{q^2}} &= q^{(z-1/2)^2}\prod_{n\ge1}\frac{(1-q^{2n-2z})(1-q^{2n+2z-2})}{(1-q^{2n-1})^2}\\
&= iq^{z^2}\,\frac{\vartheta_1(iz\ln q)}{\vartheta_4} = -\sin_q \pi(z+1)\\
&= \frac{\displaystyle\sum_{-\infty<n<\infty}(-1)^n q^{(n-z+\frac12)^2}}{\displaystyle\sum_{-\infty<n<\infty}(-q)^{n^2}}
\end{aligned}$$

and

$$\begin{aligned}
\cos_q \pi z := \sin_q \pi(\tfrac12 - z) &= q^{z^2}\prod_{n\ge1}\frac{(1-q^{2n-2z-1})(1-q^{2n+2z-1})}{(1-q^{2n-1})^2}\\
&= q^{z^2}\,\frac{\vartheta_4(iz\ln q)}{\vartheta_4} = \frac{\displaystyle\sum_{-\infty<n<\infty}(-1)^n q^{(n-z)^2}}{\displaystyle\sum_{-\infty<n<\infty}(-q)^{n^2}}.
\end{aligned}$$

While Π_q is expressible with two Dedekind η functions, Jacobi's *æquatio identica satis abstrusa* ([5]) returns the favor:

$$\eta(q) := q^{1/24}\prod_{n\ge1}1-q^n = \left(\frac{\Pi_q^5}{\Pi_{q^2}^2} - 16\,\Pi_q\,\Pi_{q^2}^2\right)^{1/6}.$$

So we see that $\sin_q \pi z$ and $\cos_q \pi z$ are period 2, unit amplitude functions with many of the properties of their $q \to 1$ ancestors:

$$\begin{aligned}
\sin_q(x+y) &= \frac{\sin_q(x-y)}{\sin_{q^2}(x-y)}(\sin_{q^2} x \cos_{q^2} y + \cos_{q^2} x \sin_{q^2} y)\\
\cos_q(x+y) &= \frac{\cos_q(x-y)}{\cos_{q^2}(x-y)}(\cos_{q^2} x \cos_{q^2} y - \sin_{q^2} x \sin_{q^2} y)\\
\cos_q 2z &= (\cos_{q^2} z)^2 - (\sin_{q^2} z)^2 = \cos 2z \prod_{n\geq 0}(\sin^2_{q^{2^{-n}}} z + \cos^2_{q^{2^{-n}}} z)\\
&= (\cos_q z)^4 - (\sin_q z)^4\\
\sin_q 2z &= \frac{\Pi_q}{\Pi_{q^2}} \sin_{q^2} z \cos_{q^2} z\\
&= \frac{1}{2}\frac{\Pi_q}{\Pi_{q^4}}\sqrt{(\sin_{q^4} z)^2 - (\sin_{q^2} z)^4} = \frac{1}{2}\frac{\Pi_q}{\Pi_{q^4}}\sqrt{(\cos_{q^4} z)^2 - (\cos_{q^2} z)^4}\\
\sin_q 3z &= \frac{\Pi_q}{\Pi_{q^3}}(\cos_{q^3} z)^2 \sin_{q^3} z - (\sin_{q^3} z)^3\\
&= \frac{1}{3}\frac{\Pi_q}{\Pi_{q^9}} \sin_{q^9} z - \left(1 + \frac{1}{3}\frac{\Pi_q}{\Pi_{q^9}}\right)(\sin_{q^3} z)^3\\
\sin_q 5z &= \frac{\Pi_q}{\Pi_{q^5}}(\cos_{q^5} z)^4 \sin_{q^5} z - \sqrt{\frac{\Pi_q^3}{\Pi_{q^5}^3} - 2\frac{\Pi_q^2}{\Pi_{q^5}^2} + 5\frac{\Pi_q}{\Pi_{q^5}}}\,(\cos_{q^5} z)^2(\sin_{q^5} z)^3\\
&\qquad + (\sin_{q^5} z)^5
\end{aligned}$$

$$\begin{aligned}
&\sin_q(x-w)\sin_q(x-y)\sin_q(y-w)\sin_q(y-z)\sin_q(z-w)\sin_q(z-x)\\
&+\cos_q(x-w)\cos_q(x-y)\sin_q(y-w)\cos_q(y-z)\cos_q(z-w)\sin_q(z-x)\\
&+\cos_q(x-w)\sin_q(x-y)\cos_q(y-w)\cos_q(y-z)\sin_q(z-w)\cos_q(z-x)\\
&+\sin_q(x-w)\cos_q(x-y)\cos_q(y-w)\sin_q(y-z)\cos_q(z-w)\cos_q(z-x) = 0
\end{aligned}$$

I actually discovered this last formula in q-land, via a sequence of symmetrizing generalizations guided by the additional structure imposed by the qs.

Strip mining over undetermined linear combinations of extensive Taylor expansions turned up the empirical relations

$$\sum_{n\ge1}\frac{q^n}{(1-q^n)^2}-2\sum_{n\ge1}\frac{q^{2n}}{(1-q^{2n})^2}=\frac{1}{24}\Big(\frac{\Pi_q^4}{\Pi_{q^2}^2}-1\Big)+\frac{2}{3}\,\Pi_{q^2}^2$$

$$\sum_{n\ge1}\frac{q^n}{(1-q^n)^2}-3\sum_{n\ge1}\frac{q^{3n}}{(1-q^{3n})^2}=\frac{(\Pi_q^2+3\,\Pi_{q^3}^2)^2}{12\,\Pi_q\,\Pi_{q^3}}-\frac{1}{12}$$

$$\sum_{n\ge1}\frac{q^n}{(1-q^n)^2}-4\sum_{n\ge1}\frac{q^{4n}}{(1-q^{4n})^2}=\frac{1}{8}\Big(\frac{\Pi_q^4}{\Pi_{q^2}^2}-1\Big)$$

$$\sum_{n\ge1}\frac{q^{2n}}{(1-q^{2n})^2}-9\sum_{n\ge1}\frac{q^{18n}}{(1-q^{18n})^2}=\frac{\Pi_{q^3}^3}{\Pi_q}+\frac{1}{3}\Big(\frac{\Pi_{q^3}^3}{\Pi_{q^9}}-1\Big)$$

$$\sum_{n\ge1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-2\sum_{n\ge1}\frac{q^{4n-2}}{(1-q^{4n-2})^2}=\Pi_{q^2}^2=\sum_{n\ge1}\frac{(2n-1)q^{2n-1}}{1-q^{4n-2}}$$

$$\sum_{n\ge1}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-5\sum_{n\ge1}\frac{q^{10n-5}}{(1-q^{10n-5})^2}=\Pi_{q^5}^2\,\frac{\dfrac{\Pi_{q^5}^2}{\Pi_{q^{10}}^2}+16\dfrac{\Pi_{q^{10}}^2}{\Pi_{q^5}^2}}{\dfrac{\Pi_q}{\Pi_{q^5}}-4-\dfrac{\Pi_{q^5}}{\Pi_q}}.$$

$$\sum_{n\ge0}\frac{1-q^{12n+1}}{1-q^2}\;\frac{(2n-\frac{3}{2})!_{q^2}(2n+\frac{1}{2})!_{q^2}}{(4n+1)!_{q^2}}q^{4n^2}=-q^{1/4}\sqrt{\pi_q\pi_{q^5}\frac{1-q^2}{1-q^{10}}}$$

This last is the $a\to0$ case of the summation identity of Section **1**. Notice how much easier it is to take the $q\to1$ limit in this form than in that.

A limiting case of a certain two parameter path invariance result (omitted) can be written

$$\sum_{k\ge0}\frac{(a+k-1)!_q(k-a)!_q}{(2k)!_q}q^{k^2}=\frac{\pi_{q^{1/2}}}{\sin_{q^{1/2}}\pi a}\,\frac{\sin_{q^{3/2}}\pi\frac{a+1}{3}}{\sin_{q^{3/2}}\frac{\pi}{3}}\;q^{(a-1)a/3}.$$

Although Professor Andrews has discovered an equivalent result in Ramanujan's "lost" "notebook" ([6]), it is worth noting how the q-trigonometric version clearly reveals this $f(a)$ as period 2 poles times period 6 zeros times a quadratic power of q.

Professors Andrews and Berndt also inform me that J. W. L. Glaisher had at least the first half of the identity

(Glaisher) $$\Pi_q^4=\sum_{n\ge1}\frac{n^3q^n}{1-q^{2n}}=\sum_{n\ge1}\mathrm{Li}_{-3}(q^{2n-1})$$

by 1905 ([7]). Combinatorially, it says that the number of ways to express $n-1$ as the sum of 8 triangular numbers equals n^3 times the sum of the cubes of the reciprocals of the odd divisors of n. For an explanation of the Li_{-3}, see section 12.

11. From the q-Land Chamber of Commerce

I found the (Glaisher) identity simply by letting the angles vanish in a q-trig identity. This illustrates one of the several advantages of q-identities: even after giving back all the conventional parameters, you may still have a neat result.

I have already mentioned that q-structures led me to the four angle trig identity.

Yet another advantage is that you can test q-identities merely by Taylor expanding with respect to q, whereas in $q \to 1$ land, you are unlikely to be able to expand with respect to any of the remaining parameters.

Computer algebra has been of great help in exploring q-land, but now perhaps q-land can return the favor, by deflating the intermediate expression swell that burdens many rational function manipulations.

For example, the path-invariance check for the matrices underlying Dougall's theorem requires the expansion of the numerator of expression (D2), below,

$$\text{(D1)}\qquad \frac{q^{-n+k-i-h-g+1}\ (1-q^{n+g})\ \left(1-q^{n+h}\right)\ \left(1-q^{n+i}\right)\ \left(1-q^{n+2k-i-h-g+2}\right)}{\left(1-q^{k-i-h-g}\right)\ \left(1-q^{n+k-g+1}\right)\ \left(1-q^{n+k-h+1}\right)\ \left(1-q^{n+k-i+1}\right)} - \frac{\left(1-q^{k-h-g+1}\right)\ \left(1-q^{k-i-g+1}\right)\ \left(1-q^{k-i-h+1}\right)\ \left(1-q^{2n+k+1}\right)}{\left(1-q^{k-i-h-g}\right)\ \left(1-q^{n+k-g+1}\right)\ \left(1-q^{n+k-h+1}\right)\ \left(1-q^{n+k-i+1}\right)}$$

(C2) `COMBINE(QONE(D1)); /*` $\lim_{q\to 1}$ of above `*/`

$$\text{(D2)}\qquad \frac{(n+g)(n+h)(n+i)(n+2k-i-h-g+2)-(k-h-g+1)(k-i-g+1)(k-i-h+1)(2n+k+1)}{(k-i-h-g)(n+k-g+1)(n+k-h+1)(n+k-i+1)}$$

which results in the 99 term expression (D3).

(C3) `EXPAND(NUM(D2));`

(D3) $n^4+2kn^3+2n^3+2ikn^2+2hkn^2+2gkn^2-i^2n^2-hin^2-gin^2+2in^2-h^2n^2-ghn^2+2hn^2-g^2n^2+2gn^2-2k^3n+4ik^2n+4hk^2n+4gk^2n-6k^2n-2i^2kn-4hikn-4gikn+8ikn-2h^2kn-4ghkn+8hkn-2g^2kn+8gkn-6kn+hi^2n+gi^2n-2i^2n+h^2in+2ghin-4hin+g^2in-4gin+4in+gh^2n-2h^2n+g^2hn-4ghn+4hn-2g^2n+4gn-2n-k^4+2ik^3+2hk^3+2gk^3-4k^3-i^2k^2-3hik^2-3gik^2+6ik^2-h^2k^2-3ghk^2+6hk^2-g^2k^2+6gk^2-6k^2+hi^2k+gi^2k-2i^2k+h^2ik+4ghik-6hik+g^2ik-6gik+6ik+gh^2k-2h^2k+g^2hk-6ghk+6hk-2g^2k+6gk-4k-ghi^2+hi^2+gi^2-i^2-gh^2i+h^2i-g^2hi+4ghi-3hi+g^2i-3gi+2i+gh^2-h^2+g^2h-3gh+2h-g^2+2g-1$

(C4) `LENGTH(D3);`

(D4) $$99$$

But (D2) is the $q \to 1$ limit of (D1), from the q version of Dougall's theorem, known as Jackson's. Even though (D1) is more complicated than (D2), its expansion, (D5), has fewer than 1/6 as many terms!

(C5) `EXPAND(NUM(COMBINE(D1)));`

(D5) $$q^{3n+3k-i-h-g+3} - q^{2n+4k-2i-2h-2g+4} - q^{2n+2k-i-h+2} - q^{2n+2k-i-g+2} - q^{2n+2k-h-g+2} + q^{n+3k-i-2h-2g+3} + q^{n+3k-2i-h-2g+3} + q^{n+3k-2i-2h-g+3} + q^{n+k-i+1} + q^{n+k-h+1} + q^{n+k-g+1} + q^{-n+k-i-h-g+1} - q^{2k-i-h-2g+2} - q^{2k-i-2h-g+2} - q^{2k-2i-h-g+2} - 1$$

(C6) `LENGTH(D5);`

(D6) $$16$$

This is because the parameters in the exponents can only combine additively, and are therefore immune to the profusion of cross-terms plaguing ordinary polynomial multiplication.

And yet, if you needed it, the 99 term expansion is recoverable from the 16 q-terms via four applications of l'Hospital's rule. But fortunately, (D5) is only an intermediate result, whose conversion can be postponed the until it's trivial.

12. Commercial for negapolylogs

(If Sesame Street can run commercials for letters of the alphabet, I demand equal time for my favorite special functions.) Running the polylog recurrence in reverse, we differentiate our way back through a sequence of rational functions which generate the Eulerian number triangle:

$$\mathrm{Li}_0(x) = x\frac{1}{1-x}$$

$$\mathrm{Li}_{-1}(x) = x\frac{1}{(1-x)^2}$$

$$\mathrm{Li}_{-2}(x) = x\frac{x+1}{(1-x)^3}$$

$$\mathrm{Li}_{-3}(x) = x\frac{x^2+4x+1}{(1-x)^4}$$

$$\mathrm{Li}_{-4}(x) = x\frac{x^3+11x^2+11x+1}{(1-x)^5}$$

$$\vdots$$

These are *useful* rational functions. While yet retaining certain loggish tendencies:

$$\begin{aligned}\mathrm{Li}_{-k}(e^z) &= \frac{d^k}{dz^k}\,\mathrm{Li}_0(e^z) \qquad\qquad k \geq 0\\ &= \frac{k!}{(-z)^{k+1}} - \sum_{n\geq 0}\frac{B_{n+k+1}(1)}{n+k+1}\frac{z^n}{n!}.\end{aligned}$$

(Note the gap of k powers of z.) The generating function is

$$\sum_{k\geq 0}\frac{\mathrm{Li}_{-k}(e^{-z})}{k!}t^k = \frac{1}{e^{z-t}-1}$$

A sample application is the expansion of tan (or sec or cot or …) about an arbitrary point,

$$\frac{d^k}{dx^k}\tan x = (2i)^{k+1}\,\mathrm{Li}_{-k}(-e^{2ix}),$$

which would otherwise invoke misleadingly transcendental polygammas. And here is the nub of the "q-Stirling" formula

$$\log\prod_{n\geq z+y} 1-q^n \ \sim_z \ \sum_{k\geq 0}\frac{B_k(y)}{k!}(\ln q)^{k-1}\,\mathrm{Li}_{2-k}(q^z).$$

And we can use negapolylogs to compute Bernoulli numbers and special values of the Euler polynomials.

$$
\begin{aligned}
\mathrm{Li}_{-k}(-1) &= (-1)^k(2^{k+1}-1)\frac{B_{k+1}}{k+1} \;=\; \frac{(-1)^{k+1}}{2}E_k(0) = -\frac{E_k(1)}{2} \\
\mathrm{Li}_{-k}(i) &= -2^{k-1}E_k(1) + \frac{i}{2}\,E_k \\
\mathrm{Li}_{-k}(e^{\frac{\pi i}{3}}) &= \frac{1}{4}\Big((1-3^{k+1})E_k(1) + i\;3^{k+\frac{1}{2}}\big(1+(-1)^k\big)E_k(\tfrac{1}{3})\Big) \\
\mathrm{Li}_{-k}(e^{\frac{2\pi i}{3}}) &= \frac{1}{4}\Big(\frac{1-3^{k+1}}{2^{k+1}-1}E_k(1) + i\;3^{k+\frac{1}{2}}\,\frac{1+(-1)^k}{2^{k-1}+1}E_k(\tfrac{1}{3})\Big)
\end{aligned}
$$

Finally, do not confuse these with negapolygammas, ψ_{-k}, which become "more transcendental" (deeply nested integrals) with increasing k, and are effectively the logarithmic derivatives of the "higher factorials" $1^{1^k}2^{2^k}\ldots$. But that's another story.

ω. References

[1] R. Wm. Gosper, *Decision procedure for indefinite hypergeometric summation*, Proceedings of the National Academy of Sciences (USA) 75, No. 1 (January 1978), 40-42.

[2] W. Heymann, *Ueber hypergeometrische Functionen, deren letztes Element speciell ist, nebst einer Anwendung auf Algebra*, Jahresbericht der Technischen Staatslehrenstalten in Chemnitz 1897-1898, (1898), 3-44.

[3] W. Heymann, *Über hypergeometrische Funktionen, deren letztes Element speziell ist*, Z. Math. Phys. 44, (1899), 280-288.

[4] G. Andrews, *On the q-Analog of Kummer's Theorem and Applications*, Duke Mathematical Journal 40, No. 3 (September 1973), 525-528.

[5] E. T. Whittaker and G. N. Watson, (A Course of) Modern Analysis, (1902-1984), 470.

[6] G. Andrews, *Ramanujan's "Lost" Notebook II*, Adv. in Math. 41, (1981), 173-185 (Eq. $(1.10.)_R$ of Thm. 1).

[7] J. W. L. Glaisher, *Representations of a number as a sum of four squares*, Quarterly J. of Pure and Applied Math. 36 (1905), 355.